HEREDITARILY NORMAL WIJSMAN HYPERSPACES ARE METRIZABLE*

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Dedicated to Professor Mitrofan Choban and Professor Stoyan Nedev for their 70 birthday.

Abstract. In this paper, we study normality and metrizability of Wijsman hyperspaces. We show that every hereditarily normal Wijsman hyperspace is metrizable. This provides a partially answer to a problem of Di Maio and Meccariello in 1998.

1. Introduction

Throughout this paper, 2^X denotes the family of all nonempty closed subsets of a given topological space X. For a metric space (X,d), let $d(x,A) = \inf\{d(x,a) : a \in A\}$ denote the distance between a point $x \in X$ and a nonempty subset A of (X,d), and $S_d(A,\varepsilon) = \{x \in X : d(x,A) < \varepsilon\}$. A net $\{A_\alpha : \alpha \in D\}$ in 2^X is said to be *Wijsman convergent to* some A in 2^X if $d(x,A_\alpha) \to d(x,A)$ for every $x \in X$. The Wijsman topology on 2^X induced by d, denoted by $\tau_{w(d)}$, is the weakest topology such that for every $x \in X$, the distance functional $d(x,\cdot)$ is continuous. To see the structure of this topology, for any $E \subseteq X$, let $E^- = \{A \in 2^X : A \cap E \neq \emptyset\}$. It can be seen easily that the Wijsman topology on 2^X induced by d has the family

$$\left\{U^- : U \text{ is open in } X\right\} \cup \left\{\{A \in 2^X : d(x,A) > \varepsilon\} : x \in X, \varepsilon > 0\right\}$$

as a subbase, refer to [2]. Moreover, for a finite subset $E \subseteq X$ and $\varepsilon > 0$, let

$$\mathcal{N}_{A,E,\varepsilon} = \left\{B \in 2^X : |d(x,A) - d(x,B)| < \varepsilon \text{ for all } x \in E\right\}.$$

Then for any $A \in 2^X$, the collection

$$\{\mathcal{N}_{A,E,\varepsilon} : E \subseteq X \text{ is finite and } \varepsilon > 0\}$$

forms a neighborhood base of A in $\tau_{w(d)}$. This type of convergence was first introduced by Wijsman in [20] for sequences of closed convex sets in Euclidean space $\mathbb{R}^n$, when he studied optimum properties of the sequential probability ratio test. It was [17] where Wijsman convergence was considered in the general framework of a metric space, and the metrizability of the Wijsman topology of a separable metric

2010 *Mathematics Subject Classification.* Primary 54E35; Secondary 54B20, 54D15.
Keywords. Embedding, hereditarily normal, hyperspace, metrizable, normal, Wijsman topology.
*This paper was initially and partially written when the first author was in a Research and Study Leave from July to December 2009, and visited the second author in August 2009. The paper was eventually completed when the two authors met and discussed at the International Conference on Topology and the Related Fields, held at Nanjing, China, 22-25 September 2012. The two authors would like to thank the School of Computing and Mathematical Sciences at the Auckland University of Technology, and the Department of Mathematics and Statistics at the University of Helsinki for their financial supports.

space was established. Since then, there has been a considerable effort to explore various topological properties of Wijsman hyperspaces. For example, Beer [1] and Costantini [8] studied Polishness of Wijsman hyperspaces, Cao and Tomita [6] as well as Zsilinszky [21] investigated Baireness of Wijsman hyperspaces, Cao and Junnila [4] studied Amsterdam properties of Wijsman spaces. However, Wijsman hyperspaces are far to be completely understood, and still there are many problems concerning fundamental properties of these objects unsolved. This motivates the authors to continue their study of Wijsman hyperspaces in the present paper.

Note that all Wijsman topologies are Tychonoff, since they are weak topologies. In a more recent paper, Cao, Junnila and Moors [5] showed that Wijsman hyperspaces are universal Tychonoff spaces in the sense that every Tychonoff space is embeddable as a closed subspace in the Wijsman hyperspace of a complete metric space which is locally $\mathbb{R}$. Thus, one of the fundamental problems is to determine when a Wijsman hyperspace is normal. The problem was first mentioned by Di Maio and Meccariello in [9], where it was asked whether the normality of a Wijsman hyperspace is equivalent to its metrizability. A partial solution to this problem, which asserts that the answer is "yes" when the base space of a Wijsman hyperspace is a normed linear space, was recently observed by Holá and Novotný in [13]. The main purpose of this paper is to give another partial answer to this problem. By using techniques similar to those of Keesling in [15], we are able to establish that a Wijsman hyperspace is hereditarily normal if and only if it is metrizable.

The rest of this paper is organized as follows. In Section 2, an overview on the normality and metrizability of basic types of hyperspaces is provided. The main result and its proof are given in Section 3. Our terminology and notation are standard. For undefined terms, refer to [2], [3] or [10].

2. Normality and metrizability of hyperspaces

It has been an interesting and challenging problem in general topology to characterize normality of the hyperspace of a topological space. In 1955, Ivanova [14] showed that $2^{\mathbb{N}}$ with the Vietoris topology is not normal, where $\mathbb{N}$ is equipped with the discrete topology. Continuing in this direction, Keesling [15] proved that under the CH (Continuum Hypothesis), for a Tychonoff space X, $(2^X, \tau_V)$ is normal if and only if $(2^X, \tau_V)$ is compact (and thus if and only if X is compact), where τ_V denotes the Vietoris topology on 2^X. In addition, he also showed in [16] that for a regular T_1 space X, a number of covering properties of $(2^X, \tau_V)$ (including Lindelöfness, paracompactness, metacompactness, and meta-Lindelöfness) are equivalent to compactness of $(2^X, \tau_V)$. Finally, Veličko [19] further showed that Keesling's result on normality also holds without the CH. This completely solved the normality problem of Vietoris hyperspaces. The normality problem of Fell hyperspaces was settled by Holá, Levi and Pelant in [12], where they showed that $(2^X, \tau_F)$ is normal if and only if $(2^X, \tau_F)$ is Lindelöf, if and only if X is locally compact and Lindelöf, here τ_F denotes the Fell topology on 2^X. Since in general the Wijsman topology induced by a metric is coarser than the Vietoris topology but finer than the Fell topology induced by the same metric, the following natural question arises.

Problem 2.1. *Let (X, d) be a metric space. When is the Wijsman hyperspace $\big(2^X, \tau_{w(d)}\big)$ a normal space?*

Let us temporarily put the normality problem of Wijsman hyperspaces aside. Instead, let us recall when a hyperspace is metrizable. A classical result claims that for a T_1 space X, $(2^X, \tau_V)$ is metrizable if and only if X is compact and metrizable, refer to [10, p.298]. A corresponding result for the Fell topology states that, for a Hausforff space X, $(2^X, \tau_F)$ is metrizable if and only if X is locally compact and second countable (and thus Lindelöf), refer to [11]. For Wijsman hyperspaces, we have the following classical result.

Theorem 2.2 ([17]). *Let (X, d) be a metric space. Then $\left(2^X, \tau_{w(d)}\right)$ is metrizable if and only if (X, d) is separable.*

Indeed, if $\{x_n : n \in \mathbb{N}\}$ is any dense subset of X, then it can be checked that

$$\varrho_d(A, B) = \sum_{n=1}^{\infty} \frac{|d(x_n, A) - d(x_n, B)| \wedge 1}{2^n}$$

defines a metric on 2^X that is compatible with $\tau_{w(d)}$. As a consequence of this result, $\left(2^X, \tau_{w(d)}\right)$ is Lindelöf if and only if it is metrizable. Note that for any metric space (X, d), we have that $\left(2^X, \tau_{w(d)}\right)$ is countably compact if and only if $\left(2^X, \tau_{w(d)}\right)$ is compact. Theorem 3.5 in [17] also claims that $\left(2^X, \tau_{w(d)}\right)$ is metrizable if and only if $\{X\}$ is a G_δ-point of $\left(2^X, \tau_{w(d)}\right)$. As a consequence, the metrizability of $\left(2^X, \tau_{w(d)}\right)$ is equivalent to a large number of generalized metric properties. For example, $\left(2^X, \tau_{w(d)}\right)$ is metrizable if and only if it has a G_δ-diagonal or it is semi-stratifiable.

In the light of the work of Keesling in [16] and Veličko in [19] on the normality of Vietoris hyperspaces as well as the work of Holá et al. on the normality of Fell hyperspaces, one may wonder whether the paracompactness and the metrizability of Wijsman hyperspaces are equivalent. These facts motivated Di Maio and Meccariello to pose the following natural problem in 1998, which also brings the normality and the metrizability of Wijsman hyperspaces together.

Problem 2.3 ([9]). *It is known that if (X, d) is a separable metric space, then $\left(2^X, \tau_{w(d)}\right)$ is metrizable and so paracompact and normal. Is the opposite true? Is $\left(2^X, \tau_{w(d)}\right)$ normal if, and only if, $\left(2^X, \tau_{w(d)}\right)$ is metrizable?*

Note that neither Problem 2.1 nor Problem 2.3 is completely solved. An affirmative solution to Problem 2.3 would also solve Problem 2.1. The following partial answer to Problem 2.3 was recently established by Holá and Novotný in [13].

Theorem 2.4 ([13]). *Let $(X, \|\cdot\|)$ be a normed linear space, and let d be the metric on X induced by the norm $\|\cdot\|$. Then $\left(2^X, \tau_{w(d)}\right)$ is normal if and only if it is metrizable.*

Given a topological space X, define $\mathrm{nlc}(X)$ by

$$\mathrm{nlc}(X) = \{x \in X : \ x \text{ has no compact neighbourhood in } X\}.$$

The following result was established by Chaber and Pol in [7].

Theorem 2.5 ([7]). *If (X, d) is a metric space such that $\mathrm{nlc}(X)$ is non-separable, then $\left(2^X, \tau_{w(d)}\right)$ contains a closed copy of $\mathbb{N}^{\omega_1}$.*

Since $\mathbb{N}^{\omega_1}$ is non-normal, as a corollary of Theorem 2.5, if (X, d) is a metric space such that $\mathrm{nlc}(X)$ is non-separable, then $\left(2^X, \tau_{w(d)}\right)$ is non-normal. In particular, if $(X, \|\cdot\|)$ is a non-separable normed linear space and d is the metric on X induced

by the norm $\|\cdot\|$, then $(2^X, \tau_{w(d)})$ is non-normal. Therefore, Theorem 2.4 can be viewed as a corollary of Theorem 2.5.

3. The main result

In this section, we shall prove the following main result of this paper, which can be treated as a partial answer to Problem 2.1 and Problem 2.3.

Theorem 3.1. *Let (X, d) be a metric space. The following are equivalent.*

(i) $(2^X, \tau_{w(d)})$ *is metrizable.*
(ii) $(2^X \setminus \{X\}, \tau_{w(d)})$ *is paracompact.*
(iii) $(2^X, \tau_{w(d)})$ *is hereditarily normal.*

To prove the above theorem, we use the embedding techniques, similar to those used by Keesling in [15]. In what follows, the ordinals ω_1 and $\omega_1 + 1$ are viewed as topological spaces equipped with the order topology.

Proposition 3.2. *Let (X, d) be a non-separable metric space. Then for any $n \geq 1$, the Wijsman hyperspace $(2^X, \tau_{w(d)})$ contains a copy of $(\omega_1 + 1)^n$.*

Proof. Let $Y_n = (\omega_1 + 1)^n$. Since (X, d) is non-separable, there exist $\varepsilon > 0$ and a set $D \subseteq X$ with $|D| = \aleph_1$ which is ε-discrete, that is, $d(x, y) \geq \varepsilon$ for all distinct $x, y \in D$. Let $n \geq 1$. We express D as the disjoint union $D = \bigcup_{i=0}^{n} D_i$ such that $D_0 = \{d\}$ and $|D_i| = \aleph_1$ for all $1 \leq i \leq n$. Next, for $1 \leq i \leq n$, we enumerate D_i as $D_i = \{x^i_\alpha : \alpha < \omega_1\}$, and for each $\alpha \leq \omega_1$, we put $L^i_\alpha = \{x^i_\lambda \in D_i : \lambda < \alpha\}$. Obviously, each L^i_α is closed in X. Define a mapping $\varphi : Y_n \to (2^X, \tau_{w(d)})$ by the formula $\varphi(\alpha_i) = D_0 \cup \bigcup_{i=1}^{n} L^i_{\alpha_i}$. It is clear that φ is one-to-one.

To see that φ is continuous, suppose $\varphi(\alpha_i) \cap V \neq \emptyset$ for some open set $V \subseteq X$. If $D_0 \cap V \neq \emptyset$, there is nothing to verify. So, we assume that $L^i_{\alpha_i} \cap V \neq \emptyset$ for some $1 \leq i \leq n$. Hence, there is some non-limit ordinal $\lambda_i < \alpha_i$ such that $L^i_{\lambda_i} \cap V \neq \emptyset$. It follows that for any neighborhood $N(\alpha_j)$ of α_j with $j \neq i$, we have

$$\varphi\left(\prod_{j<i} N(\alpha_j) \times (\lambda_i, \alpha_i] \times \prod_{j>i} N(\alpha_j)\right) \subseteq V^-.$$

On the other hand, if $d(x, \varphi(\alpha_i)) > r$ for some $x \in X$ and $r > 0$, then for any $\lambda_i \leq \alpha_i$ we have $d(x, \varphi(\lambda_i)) > r$. Thus, we have verified that φ is continuous at any point $(\alpha_i) \in Y_n$.

Since Y_n is compact, the continuous one-to-one mapping φ is an embedding. $\square$

Corollary 3.3. *Let (X, d) be a metric space. Then for the space $(2^X, \tau_{w(d)})$, metrizability is equivalent to each one of the following properties: Fréchetness, sequentiality, countable tightness.*

Proof. Proposition 3.2 shows, in particular, that if (X, d) is non-separable, then $(2^X, \tau_{w(d)})$ contains a copy of ω_1+1. Since ω_1+1 does not have countable tightness, the conclusion follows. $\square$

Question 3.4. *Let (X, d) be a non-separable metric space. Can $(2^X, \tau_{w(d)})$ contain a copy of $(\omega_1 + 1)^\omega$ or $(\omega_1 + 1)^{\omega_1}$?*

For the proof of our next proposition, we need an auxiliary result.

Lemma 3.5. *Let (X,d) be a metric space. If $\mathcal{F}$ is a directed family in 2^X such that $H=\bigcup\mathcal{F}$ is closed, then H is a limit point of the net $(\mathcal{F},\subseteq)$ in $\left(2^X,\tau_{w(d)}\right)$.*

Proof. The proof of this lemma is straightforward, and thus is omitted. □

In his proof of the equivalence of normality and compactness for Vietoris hyperspaces (under the CH), Keesling [15] established that, for a noncompact Tychonoff space X, $(2^X,\tau_V)$ contains a closed copy of the space $\omega_1\times(\omega_1+1)$. We are not able to obtain a similar embedding in the Wijsman hyperspace of a non-separable metric space (X,d). However, we have the following result.

Proposition 3.6. *Let (X,d) be a non-separable metric space. Then the subspace $2^X\setminus\{X\}$ of $\left(2^X,\tau_{w(d)}\right)$ contains a closed copy of the space $\omega_1\times(\omega_1+1)$.*

Proof. Since (X,d) is non-separable, there exist $\varepsilon>0$ and an ε-discrete proper subset $D=\{x_{\alpha,\beta}:\alpha<\omega_1 \text{ and } \beta\le\omega_1\}$ of X, with $x_{\alpha,\beta}\neq x_{\alpha',\beta'}$ for $(\alpha,\beta)\neq(\alpha',\beta')$. For every $\alpha<\omega_1$, let $D_\alpha=\{x_{\alpha,\beta}:\beta\le\omega_1\}$ and $G_\alpha=S_d(D_\alpha,\frac{\varepsilon}{4})$. For every $\alpha<\omega_1$ and each $\beta\le\omega_1$, let

$$F_\alpha=X\setminus\left(\bigcup_{\gamma\ge\alpha}G_\gamma\right)$$

and

$$S_\beta=\{x_{\gamma,\delta}:\gamma<\omega_1 \text{ and } \delta<\beta\}.$$

Note that the families $\mathcal{F}=\{F_\alpha:\alpha<\omega_1\}$ and $\mathcal{S}=\{S_\beta:\beta\le\omega_1\}$ are "continuously increasing", in the sense that $F_\alpha=\bigcup\{F_{\gamma+1}:\gamma<\alpha\}$ and $S_\beta=\bigcup\{S_{\delta+1}:\delta<\beta\}$ for all $0<\alpha<\omega_1$ and $0<\beta\le\omega_1$. We show that the subspace

$$\mathcal{H}=\{F_\alpha\cup S_\beta:\alpha<\omega_1 \text{ and } \beta\le\omega_1\}$$

of $(2^X,\tau_{w(d)})$ is homeomorphic to the product space $\omega_1\times(\omega_1+1)$. Define a mapping $\varphi:\omega_1\times(\omega_1+1)\to\mathcal{H}$ by the formula $\varphi(\alpha,\beta)=F_\alpha\cup S_\beta$, and note that φ is one-to-one and onto.

To show that φ is continuous, let $A\subseteq\omega_1\times(\omega_1+1)$, and let $(\alpha,\beta)\in\overline{A}$. We show that $\varphi(\alpha,\beta)\in\overline{\varphi(A)}$. Let $A'=\{(\gamma,\delta)\in A:\gamma\le\alpha \text{ and } \delta\le\beta\}$, and note that $(\alpha,\beta)\in\overline{A'}$. Note that, for all $(\gamma,\delta),(\gamma',\delta')\in A'$, there exists $(\mu,\nu)\in A'$ such that $\mu\ge\max(\gamma,\gamma')$ and $\nu\ge\max(\delta,\delta')$. As a consequence, the family $\{F_\gamma\cup S_\delta:(\gamma,\delta)\in A'\}$ is directed. Since $(\alpha,\beta)\in\overline{A'}$, we have for all $\alpha'<\alpha$ and $\beta'<\beta$ that there exists $(\gamma,\delta)\in A'$ such that $\gamma\ge\alpha'$ and $\delta\ge\beta'$. As $\mathcal{F}$ and $\mathcal{S}$ are continuously increasing, it follows that

$$\bigcup\{F_\gamma\cup S_\delta:(\gamma,\delta)\in A'\}=F_\alpha\cup S_\beta.$$

From the foregoing it follows by Lemma 3.5 that the net $(\{F_\gamma\cup S_\delta:(\gamma,\delta)\in A'\},\subseteq)$ converges to $F_\alpha\cup S_\beta$ in $\tau_{w(d)}$. As a consequence, $\varphi(\alpha,\beta)\in\overline{\varphi(A')}\subseteq\overline{\varphi(A)}$. We have shown that φ is continuous.

Next, we show that φ is open. Let W be an open subset of $\omega_1\times(\omega_1+1)$. To show that $\varphi(W)$ is open in $\mathcal{H}$, let $(\alpha,\beta)\in W$. Denote by J the element $\varphi(\alpha,\beta)=F_\alpha\cup S_\beta$ of the set $\varphi(W)$. There exist $\gamma<\alpha$ and $\delta<\beta$ such that $(\gamma,\alpha]\times(\delta,\beta]\subseteq W$. Let $E=\{x_{\alpha,\beta},x_{\alpha,\delta},x_{\gamma,\beta}\}$ and

$$\mathcal{N}_{J,E,\varepsilon/2}=\left\{H\in\mathcal{H}:|d(x,H)-d(x,J)|<\frac{\varepsilon}{2} \text{ for every } x\in E\right\}.$$

Note that $\mathcal{N}_{J,E,\varepsilon/2}$ is a neighborhood of J in $\mathcal{H}$. We show that $\mathcal{N}_{J,E,\varepsilon/2} \subseteq \varphi(W)$. Let $H \in \mathcal{N}_{J,E,\varepsilon/2}$, and let $\mu < \omega_1$ and $\nu \leq \omega_1$ be such that $H = F_\mu \cup S_\nu$. To show that $H \in \varphi(W)$, we need to show that the inequalities $\gamma < \mu \leq \alpha$ and $\delta < \nu \leq \beta$ hold. For the element $x_{\alpha,\beta}$ of E, we have $x_{\alpha,\beta} \in G_\alpha \subseteq X \setminus F_\alpha$ and $x_{\alpha,\beta} \notin S_\beta$. It follows that $x_{\alpha,\beta} \notin J$ and hence that $d(x_{\alpha,\beta}, J) \geq \varepsilon$. As a consequence,

$$d(x_{\alpha,\beta}, H) \geq d(x_{\alpha,\beta}, J) - |d(x_{\alpha,\beta}, H) - d(x_{\alpha,\beta}, J)| \geq \varepsilon - \frac{\varepsilon}{2} > 0.$$

By the foregoing, we have that $x_{\alpha,\beta} \notin H$, and this means that $x_{\alpha,\beta} \notin F_\mu$ and $x_{\alpha,\beta} \notin S_\nu$. It follows that we have $\mu \leq \alpha$ and $\nu \leq \beta$. For the element $x_{\alpha,\delta}$ of E, we have $x_{\alpha,\delta} \in S_\beta \subseteq J$, and hence $d(x_{\alpha,\delta}, J) = 0$. It follows that

$$d(x_{\alpha,\delta}, H) = |d(x_{\alpha,\delta}, H) - d(x_{\alpha,\delta}, J)| < \frac{\varepsilon}{2}.$$

Since $H \subseteq D$ and $x_{\alpha,\delta} \in D$, it follows further, by ε-discreteness of D, that $x_{\alpha,\delta} \in H$. Since $H = F_\mu \cup S_\nu$, we have either $x_{\alpha,\delta} \in F_\mu$ or $x_{\alpha,\delta} \in S_\nu$. In the first case, since $x_{\alpha,\delta} \in D_\alpha \subseteq G_\alpha$, we would have that $\alpha < \mu$; however, we showed above that $\mu \leq \alpha$. Hence we must have that $x_{\alpha,\delta} \in S_\nu$. It follows that $\delta < \nu$. We have shown that $\delta < \nu \leq \beta$. Similarly, for the element $x_{\gamma,\beta}$ of E, we have that $x_{\gamma,\beta} \in F_\alpha \subseteq J$, and hence that $d(x_{\gamma,\beta}, J) = 0$. It follows that $d(x_{\gamma,\beta}, H) < \frac{\varepsilon}{2}$, and further, that $x_{\gamma,\beta} \in H$. As a consequence, we have either $x_{\gamma,\beta} \in F_\mu$ or $x_{\gamma,\beta} \in S_\nu$. In the second case we would have that $\beta < \nu$, but this does not hold, since we showed above that $\nu \leq \beta$. Hence we must have $x_{\gamma,\beta} \in F_\mu$, and it follows from this that $\gamma < \mu$. We have shown that $\gamma < \mu \leq \alpha$. This completes the proof of openness of φ.

We have shown that the subspace $\mathcal{H}$ of $(2^X, \tau_{w(d)})$ is homeomorphic to the space $\omega_1 \times (\omega_1 + 1)$. Note that $\bigcup \mathcal{H} = D$. As a consequence, $X \notin \mathcal{H}$. To complete the proof, we show that $\mathcal{H}$ is closed in the subspace $2^X \setminus \{X\}$ of $(2^X, \tau_{w(d)})$. Let $K \in \overline{\mathcal{H}} \setminus \mathcal{H}$. To show that $K = X$, assume on the contrary that $X \setminus K \neq \emptyset$. Let $y \in X \setminus K$. There exists $\alpha_0 < \omega_1$ such that $y \notin \bigcup_{\gamma > \alpha_0} G_\gamma$. Note that $y \in F_\alpha$ for each $\alpha > \alpha_0$. The subset $\mathcal{H}_0 = \{F_\alpha \cup S_\beta : \alpha \leq \alpha_0 \text{ and } \beta \leq \omega_1\}$ of $\mathcal{H}$ is compact, because it is homeomorphic to $[0, \alpha_0] \times [0, \omega_1]$. Since $K \in \overline{\mathcal{H}} \setminus \mathcal{H}$, it follows that $K \in \overline{\mathcal{H} \setminus \mathcal{H}_0}$. Let $r = d(y, K)$ and consider the neighborhood

$$\mathcal{M}_{K,\{y\},r} = \{H \in 2^X : |d(y, H) - d(y, K)| < r\}$$

of K in $(2^X, \tau_{w(d)})$. It follows from the foregoing, that there exist $\alpha > \alpha_0$ and $\beta \leq \omega_1$ such that $F_\alpha \cup S_\beta \in \mathcal{M}_{K,\{y\},r}$. However, now we have that $y \in F_\alpha$ and hence $d(y, F_\alpha \cup S_\beta) = 0$. Since $F_\alpha \cup S_\beta \in \mathcal{M}_{K,\{y\},r}$, we have $d(y, K) < r$; this, however, contradicts with the definition of r. We have shown that $\overline{\mathcal{H}} \setminus \mathcal{H} \subseteq \{X\}$ and hence that $\mathcal{H}$ is closed in $2^X \setminus \{X\}$. □

Corollary 3.7. *Let (X, d) be a metric space. The following are equivalent.*

(i) $(2^X, \tau_{w(d)})$ *is metrizable.*
(ii) $(2^X \setminus \{X\}, \tau_{w(d)})$ *is metacompact.*
(iii) $(2^X \setminus \{X\}, \tau_{w(d)})$ *is meta-Lindelöf.*
(iv) $(2^X \setminus \{X\}, \tau_{w(d)})$ *is orthocompact.*

Proof. We only need to show that (iv) ⇒ (i). Assume that (X, d) is not separable. By Proposition 3.6, $(2^X \setminus \{X\}, \tau_{w(d)})$ contains a closed copy of $\omega_1 \times (\omega_1 + 1)$. As $(2^X \setminus \{X\}, \tau_{w(d)})$ is orthocompact, then $\omega_1 \times (\omega_1 + 1)$ is orthocompact, which contradicts with a result of Scott in [18]. □

We now use Proposition 3.6 to prove Theorem 3.1.

Proof of Theorem 3.1. We only need to prove that (iii) implies (i). Assume that (iii) holds, but $\left(2^X, \tau_{w(d)}\right)$ is not metrizable. By Theorem 2.2, (X, d) is not separable. By Proposition 3.6, $\left(2^X \setminus \{X\}, \tau_{w(d)}\right)$ contains a closed copy of $\omega_1 \times (\omega_1 + 1)$. Since $\omega_1 \times (\omega_1 + 1)$ is not normal, (iii) does not hold. This is a contradiction. $\square$

We conclude this paper with the following open question.

Question 3.8. *Let (X, d) be a metric space. If $\left(2^X, \tau_{w(d)}\right)$ is non-normal, does $\left(2^X, \tau_{w(d)}\right)$ contain a closed copy of $\omega_1 \times (\omega_1 + 1)$?*

Note that there exists a metric space (X, d) such that $\left(2^X, \tau_{w(d)}\right)$ is non-normal, but $\left(2^X, \tau_{w(d)}\right)$ contains no closed copy of $\mathbb{N}^{\omega_1}$. Indeed, take any set X with $|X| = \omega_1$ and equip X with the 0-1 metric d. By Remark 3.1 of [7], $\left(2^X, \tau_{w(d)}\right)$ is homeomorphic to $\{0,1\}^{\omega_1} \setminus \{\mathbf{0}\}$, which is locally compact. Thus, $\left(2^X, \tau_{w(d)}\right)$ contains no closed copy of $\mathbb{N}^{\omega_1}$.

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